IUCAA GR Refresher Course Tutorials by BM

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Tutorial:2 (24-06-2025)

Problem:1 Gauge Invariance and the Photon Mass

Consider the modified Lagrangian density for the electromagnetic field with a mass term added:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor, A^μ is the four-potential, and m is a non-zero mass parameter.

Tasks:

1. Show that the mass term $\frac{1}{2}m^2 A^\mu A_\mu$ **is not invariant** under the U(1) gauge transformation:

$$A^\mu \to A^\mu + \partial^\mu \Lambda(x)$$

where $\Lambda(x)$ is an arbitrary smooth scalar function.

2. Conclude why the **photon must be massless** in standard electrodynamics.

3. In flat spacetime, the electromagnetic field strength tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and the Lagrangian density for the electromagnetic field interacting with a current is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu.$$

Tasks:

(i) When transitioning to curved spacetime, explain how the definition of the field strength tensor $F_{\mu\nu}$ changes, if at all. Does the partial derivative ∂_μ get replaced by a covariant derivative ∇_μ?

(ii) How does the Lagrangian density $\mathcal{L}$ change in curved spacetime? Write down the action integral for electrodynamics in curved spacetime and explain the role of $\sqrt{-g}$, where g is the determinant of the metric.

(iii) Comment on why $F_{\mu\nu}$ can still be written using partial derivatives in curved spacetime, even though the spacetime is not flat.

Problem:2 Exterior Derivative of the Electromagnetic Field 2-Form and Homogeneous Maxwell Equations

In differential geometry, the electromagnetic field is represented by a **2-form**:

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $A = A_\mu dx^\mu$ is the 1-form electromagnetic potential.

Let $\star F$ denote the **Hodge dual** of the 2-form F, which is also a 2-form in four-dimensional spacetime.

Tasks:

1. Show that $dF = 0$ identically, where d is the **exterior derivative**. **(Hint: Use the fact that $F = dA$, and $d^2 = 0$)**
2. Express the components of the equation $dF = 0$ and interpret them as the homogeneous Maxwell equations in tensor form:

$$\partial_{[\lambda} F_{\mu\nu]} = 0$$

3. In the language of differential geometry, the electromagnetic field is described by a 2-form:

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu,$$

and the current density is represented by a 1-form:

$$J = J_\mu\, dx^\mu.$$

Let $*F$ denote the **Hodge dual** of the 2-form F, and let $*J$ be the dual of the 1-form J, i.e., a 3-form on spacetime. Show that the **inhomogeneous Maxwell equations**,

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

can be compactly written using differential forms as:

$$d * F = *J,$$

where d is the exterior derivative, and $*$ is the Hodge star operator on the spacetime manifold.